LIE GROUPS AND LIE ALGEBRAS FINAL EXAMINATION

Total marks: 100

Attempt all questions

(1) Let G be a linear group, $\mathfrak{g}$ be its Lie algebra, and $exp : \mathfrak{g} \to G$ be the exponential map. Describe the image and the fibres of exp for the groups $SO(3)$ and $SL_2(\mathbb{R})$? (10 + 15 marks)

(2) Construct a $2 : 1$ group homomorphism $SU(2) \to SO(3)$. (10 marks)

(3) Let $f : G \to G'$ be a differentiable homomorphism between two linear groups G and G'. Let $\mathfrak{g}$ and $\mathfrak{g}'$ be their respective Lie algebras. Prove that f induces a homomorphism of Lie algebras $\phi : \mathfrak{g} \to \mathfrak{g}'$, satisfying $f \circ exp = exp \circ \phi$. (10 + 10 marks)

(4) Let G be a connected linear group, let $\mathfrak{g}$ be its Lie algebra. Let H be a connected subgroup of G and let $\mathfrak{h}$ be its Lie algebra. Prove that $\mathfrak{h}$ is an ideal of $\mathfrak{g}$, if and only if, H is a normal subgroup of G. (15 marks)

(5) Define the Ad and ad maps for a linear group G and its Lie algebra $\mathfrak{g}$, and prove that ad is the induced homomorphism of Lie algebras of Ad. (5+10 marks)

(6) Let $M = M_n(\mathbb{C})$ be the space of all $n \times n$ matrices (for some n), and let $exp : M \to M$ be the exponential map. Write down a formula for the derivative of the exponential map $d(exp)_X$ at any $X \in M$. Using this or otherwise, prove that $d(exp)_X$ is invertible, if and only if, no two eigenvalues of X differ by $2\pi ik$ for $k = 1, 2, 3, \ldots$. (5+10 marks)

Date: April 19, 2017.